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SMALL STEADY PERTURBATIONS BEHIND THE FRONT OF AN OBLIQUE SHOCK WAVE

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The effect of steady perturbations of a stepped load moving on the plane surface of a perfect medium on the hydrodynamic parameters behind an oblique shock wave front is considered. Only the case in which the system of differential equations for the perturbations is of the elliptic type is investigated on the assumption that the conditions of stability [1] of a plane stationary shock wave are satisfied. Conditions under which the solution has a logarithmic or power singularity in derivatives are discussed. The results may prove useful as a guide in numerical computations.

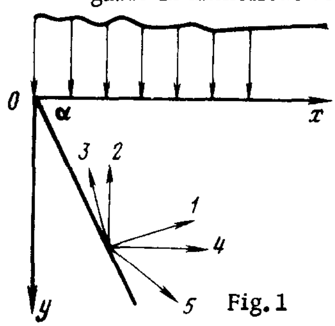

Fig. 1

1. Let us consider the stable motion of a perfect medium which fills a half-space and is subjected to a load whose profile is close to a "step" form unchanging in time. The load moves on the medium surface at constant supersonic velocity D. In the system of coordinates $\mathbf{x} = (x, y)$ attached to the load front (Fig. 1) the pressure at the boundary can be represented in the form

$$P = P_1 [1 + \varepsilon q (x)], \quad x > 0, \quad P = 0, \quad x < 0$$

It is assumed that a solution with an oblique shock wave with constant parameters behind it obtains when $\varepsilon = 0$, and that ahead of the single solution discontinuity front the medium is unperturbed. When $\varepsilon \ll 1$ we take into account in the linear formulation the effect of perturbations by representing the unknown functions $P(\mathbf{x})$ and $\mathbf{U}(\mathbf{x}) = (U, W)$ which is the velocity vector of a point of the medium determined in laboratory coordinate system, $V(\mathbf{x})$ is the specific volume, and $x = F(y)$ is the equation of the front line of the form

$$P \approx P_1 (1 + \varepsilon p), \quad U \approx U_1 (1 + \varepsilon u), \quad W \approx U_1 (-b + \varepsilon w)$$
$$V \approx V_0 (V_1 / V_0 + \varepsilon v), \quad x = F(y) \approx b [y + \varepsilon f(y)]$$

where the subscripts zero and unity denote quantities ahead and behind the front, respectively.

Conditions at the unperturbed front imply that

$$U_1 = V_0 P_1 / D, \quad b^2 = \theta_0 D^2 / (V_0 P_1) - 1 \quad (\theta_0 = 1 - V_1 / V_0)$$

We define the stable motion of perturbations within the angle α (Fig. 1) by the equations

$$(\mathbf{g}\nabla)\,\mathbf{u} = \nabla p, \quad \nabla\mathbf{u} = (1-k^2)\,\mathbf{g}\nabla p$$
$$\mathbf{u} = (u, w), \quad \mathbf{g} = (1-\theta_1, b\theta_1)\,V_0 / V_1$$
$$\theta_1 = \theta_0 / (1+b^2), \quad k^2 = 1 - [V_1 D / (V_0 c)]^2$$

where c is the speed of sound behind the front. When $b\theta_0 \ll 1$ and $V_0 / V_1 \sim 1$, these equations are considerably simpler. We have

$$\partial\mathbf{u} / \partial x = \nabla p, \quad \partial w / \partial y = -k^2 \partial p / \partial x \tag{1.1}$$

Using the laws of conservation at the discontinuity for $x = by$ we obtain the boundary conditions

$$u = p = Cf', \quad w = -b\,(1+C)\,f', \quad p = -jv \tag{1.2}$$
$$C = \frac{2b^2}{(j-1)(1+b^2)}, \quad j = \frac{P_1}{V_1 - V_0}\left(\frac{\partial V}{\partial P}\right)_H \Big|_{P=P_1}$$

where the prime and subscript H denote differentiation with respect to y and along the Hugoniot curve, respectively.

When $b\theta_0 \ll 1$ the condition at the medium surface is brought to the plane $y = 0$

$$p = q\,(x), \quad x > 0 \quad (q\,(x) \to 0, \ x \to \infty) \tag{1.3}$$

and when $x \to \infty$ and $y > 0$ the conditions become null.

We solve problem (1.1)–(1.3) on the assumption that $k^2 > 0$, i.e.

$$D_0 < D < cV_0 / V_1, \quad D_0 = (P_1 V_0 / \theta_0)^{1/2}$$

where D_0 is the shock wave velocity at point O. The system of Eqs. (1.1) is then of the elliptic kind and the problem analogous to that in [2].

2. From the first two of Eqs. (1.1) and the first of conditions (1.2) follows that $u\,(\mathbf{x}) = p\,(\mathbf{x})$. The remaining equations (1.1) are satisfied if $Q\,(z) = p + iw / k$ $(z = x + iky)$ is an analytic function of z regular inside the angle $0 < \arg z < \alpha_0$ $(\operatorname{tg}\alpha_0 = k \operatorname{tg}\alpha = k / b)$.

The real part of $Q\,(z)$ is specified on the half-line $\arg z = 0$, and on the half-line $\arg z = \alpha_0$ we have the relation between the real and imaginary parts, which in accordance with (1.2) is of the form

$$\operatorname{Re} Q = \operatorname{tg}\beta \operatorname{Im} Q, \quad \beta = \beta_0 + m\pi \tag{2.1}$$
$$\beta_0 = \operatorname{arctg}(kA / b), \quad A = C / (1+C), \quad m = 0, \ \pm 1, \pm 2, \ldots$$

where unlike in [2] $\beta_0 \neq \alpha_0$.

Function $Q\,(z)$ must vanish at infinity.

We represent the interior of angle α_0 in the upper half-plane using the conformal mapping

$$\zeta = z^\nu \quad (\nu = \pi / \alpha_0, \ \zeta = \xi + i\eta)$$

If $\eta = 0, \xi < 0$ the condition

$$\operatorname{Im} ie^{i\beta}\,Q\,(\zeta) = 0$$

which follows from (2.1) makes possible the analytic continuation into the lower half-plane through the boundary $\xi < 0, \eta = 0$. For the limit values of $Q\,(\zeta)$ above and below the axis $\xi > 0$ this condition assumes the form

$$Q_+(\xi) = e^{2i\beta} Q_-(\xi) + 2q(\xi^{1/\nu})$$

The unique solution of this problem, which is zero at infinity and bounded at the coordinate origin, is of the form

$$Q(\zeta) = \frac{\zeta^\gamma}{\pi i} \int_0^\infty \frac{q(\xi^{1/\nu})\, d\xi}{\xi^\gamma(\xi - \zeta)}, \quad \gamma = \begin{cases} 1 - \beta_0/\pi, & \beta_0 > 0 \\ -\beta_0/\pi, & \beta_0 < 0 \end{cases}$$

Reverting to the variable z we obtain

$$Q(z) = \frac{i}{\alpha_0 z^\mu} \int_0^\infty \frac{q(x)\, dx}{x^{1-\mu}(1 - x^\nu/z^\nu)}, \quad \mu = \begin{cases} \nu(1-\gamma) = \beta_0/\alpha_0, & \beta_0 > 0 \\ (\pi + \beta_0)/\alpha_0, & \beta_0 < 0 \end{cases}$$

3. It can be shown that

$$Q'(z) = \frac{i}{\alpha_0 z^{1+\mu}} \int_0^\infty \frac{x^\mu q'(x)\, dx}{1 - x^\nu/z^\nu}$$

The behavior of $Q'(z)$ when $|z| \to 0$ depends on the sign of $\nu - \mu - 1 = \nu(\gamma - 1/\nu)$. When $\gamma > 1/\nu$, then $Q'(z)$ is continuous at zero. When $\gamma = 1/\nu$ and $\gamma < 1/\nu$, function $Q'(z)$ has a logarithmic or power singularity, respectively, and the front curvature becomes infinite at point O, which indicates instability of solution in the metric $C^{(2)}$.

It can be shown that the inequality $\gamma \leqslant 1/\nu$ is equivalent to the system of inequalities $-1 \leqslant A < 0$. We shall solve that system for b^2 taking into account the conditions of stability of a plane steady shock wave [1] which can be rewritten in the form

$$-(1 + B) \leqslant j < 1 \quad (B = 2[(1 - k^2)/(1 + b^2)]^{1/2} < 2) \tag{3.1}$$

We obtain

$$b^2 \leqslant (1 - j)/(3 + j) \tag{3.2}$$

If condition (3.1) is satisfied, the quantity in the right-hand side of (3.2) is nonnegative, hence there always exists an $\alpha_* = \operatorname{arcctg}[(1 - j)/(3 + j)]^{1/2}$ such that $0 < \alpha_* < \pi/2$, and when $\alpha_* \leqslant \alpha < \pi/2$, $Q'(z)$ has a singularity. The range of such values of α widens from $j = 1$ $(\alpha_* = \pi/2)$; the slope of the Hugoniot curve at point $V = V_1$ coincides with the slope to $j = 0$ $(\alpha_* = \pi/3)$ that determines velocity D_0 the Hugoniot curve becomes vertical, and then, for negative values of j the Hugoniot curve has an anomalous run.

4. Instead of the problem of $Q(z)$ we could consider the problem of $Q'(z) = \partial p/\partial x - ik^{-1}\partial p/\partial y$. The condition for $Q'(z)$ at the front is obtained by differentiating conditions (1.2) along the front and eliminating superfluous derivatives using (1.1). It is of the form

$$\mathbf{a}\nabla p = 0, \quad \mathbf{a} = (b^2 - k^2 A, \quad b(A + 1)) = (a_1, a_2)$$

Vector $\mathbf{a}$ points out the direction of isobars emanating from the front. The condition of passing to a solution with a singularity coincides with the condition that

$a_2 = 0$. We denote the angle between vector a and the [positive] direction of the x -axis by φ. We have the following possible cases of the isobars (Fig. 1):

1) $-\pi / 2 < \varphi < 0$ for $-\infty < A < -1$ and $b^2 / k^2 < A < \infty$
2) $\varphi = -\pi / 2$ for $A = b^2 / k^2$ $(a_1 = 0)$
3) $-\pi / 2 - \alpha < \varphi < -\pi / 2$ for $0 < A < b^2 / k^2$
4) $\varphi = 0$ for $A = -1$ $(a_2 = 0)$
5) $0 < \varphi < \alpha$ for $-1 < A < 0$

This shows that $Q'(z)$ has a singularity when the isobar run is anomalous (i. e. in the cases 4) and 5)).

Note that the disclosed singularities of solution are not a consequence of linearization. It can be shown that the front curvature becomes infinite at point O within some range of external parameters, with the physical and geometrical nonlinearity taken fully into account.

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